Exact moment scaling from multiplicative noise

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For a general class of diffusion processes with multiplicative noise, describing a variety of physical as well as financial phenomena, mostly typical of complex systems, we obtain the analytical solution for the moments at all times. We allow for a nontrivial time dependence of the microscopic dynamics and we analytically characterize the process evolution, possibly toward a stationary state, and the direct relationship existing between the drift and diffusion coefficients and the time scaling of the moments.

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Many different physical phenomena exhibit a complex behavior characterized by long-range correlations, long-time memory, scale invariance, and the emergence of non Gaussian distributions associated to their statistical description. Deviations from the Maxwell-Boltzmann statistics were usually considered as a clear mark of an out of equilibrium system, but recently it has been recognized that Normality is not the most general paradigm describing the equilibrium state. Indeed, in terms of a microscopic description provided by the Langevin equation, power-law tails stem naturally assuming the damping coefficient to have a stochastic nature [1]. From a macroscopic point of view, the superposition of an additive Gaussian noise with a multiplicative one leads to a Fokker-Planck (FP) equation with linear drift and quadratic diffusion coefficients. Processes leading to a macroscopic equation with the same structure emerge in the description of several physical systems ranging from turbulent velocity flows [2], power law spectra in e^+e^-, $p\bar{p}$ and heavy ions collisions [3], anomalous diffusion phenomena [4], to the study of non stationary scaling Markov processes with Hurst exponent $H\neq 1/2$ [5]. Moreover, the same dynamics has been shown to describe heartbeat interval fluctuations, foreign exchange markets [6], option markets [7], and the statistical features of medium-term log returns in a market with both fundamental and technical traders [8]. The explicit analytical characterization of the probability density function (PDF) for these processes has been carried out only for the steady state [1], while an analytical description at finite time can be given only in terms of a formal expansion on the set of eigenfunctions of the FP operator [9]. In this Brief Report, we provide a complete description of these processes in terms of their moments, allowing for more general time dependence for both the drift and diffusion coefficients. We obtain closed-form expressions for the moments at finite time; in particular, we are able to characterize analytically how they thermalize to the stationary state and we highlight the existence of a direct, simple relationship between the function regulating the time dependence and the scaling of the process over time.

We start from the stochastic differential equation (SDE) describing the microscopic dynamics under Itô prescription [11]

$$dX_t = \frac{aX_t + b}{g(t)}dt + \sqrt{\frac{cX_t^2 + dX_t + e(t)}{g(t)}}dW_t, \tag{1}$$

with initial time condition $X_{t_0}=X_0$, $t_0\in D\subseteq[0,t_{\lim})$ with $t_{\lim}$ possibly $+\infty$; W_t is the standard Brownian motion, a, b, c, and d are real constants, $1/g(t)$ and $e(t)$ are non negative smooth functions of the time over D. For the diffusion coefficient to be meaningful, it has to satisfy $d^2-4ce(t)\leq 0$ with $c\geq 0$. Application of the Itô Lemma to $f(X_t)=X_t^n$ leads to the following integral relation

$$X_t^n = X_0^n + \int_{t_0}^{t}\frac{X_s^{n-2}}{g(s)}[F_nX_s^2 + A_nX_s + B_n(s)]ds + n\int_{t_0}^{t}\frac{X_s^{n-1}}{\sqrt{g(s)}}\sqrt{cX_s^2 + dX_s + e(s)}dW_s, \tag{2}$$

whose expectation readily provides the linear ordinary differential equation satisfied by the n-th order moment $\mu_n(t)=\langle X_t^n\rangle$ for $n\geq 1$

$$g(t)\frac{d}{dt}\mu_n(t) = F_n\mu_n(t) + A_n\mu_{n-1}(t) + B_n(t)\mu_{n-2}(t), \tag{3}$$

with boundary condition $\mu_n(t_0)=\langle X_0^n\rangle$. The coefficients read $F_n=na+\frac{1}{2}n(n-1)c$, $A_n=nb+\frac{1}{2}n(n-1)d$, $B_n(t)=\frac{1}{2}n(n-1)e(t)$, and we assume $\mu_0(t)=1$. If $\langle X_0^k\rangle$ for $k=1,\dots,n$ is a finite quantity, the smoothness of $1/g(t)$ and $e(t)$ ensures the existence of a unique solution $\mu_n(t)$ over an arbitrary interval $D'=[t_i,t_f]\subseteq D$ with $t_0\in D'$. In terms of the monotonously increasing function $\tau(t)=\int_{t_0}^{t}1/g(s)ds$, the solution reads

$$\mu_n(t) = \exp[F_n\tau(t)]\left[\langle X_0^n\rangle + \int_0^{\tau(t)}\exp(-F_n\tau_1)A_n\tilde{\mu}_{n-1}(\tau_1)d\tau_1 + \int_0^{\tau(t)}\exp(-F_n\tau_1)\tilde{B}_n(\tau_1)\tilde{\mu}_{n-2}(\tau_1)d\tau_1\right], \tag{4}$$

where $\tilde{\mu}_n(\tau)=\mu_n[t(\tau)]$ and $\tilde{B}_n(\tau)=B_n[t(\tau)]$. The previous expression lends itself to an expansion over $\langle X_0^{n-j}\rangle$, for

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$j=0,\ldots,n$ by iteratively substituting the moments entering the right-hand side with their closed-form solutions. We detail how to proceed for the simpler case $e(t)=e\geq 0$. We define type A and type B "knots" of order k whose contributions are

$$\mathbb{A}_k = A_k\int_0^\tau \exp(a_k\tau')d\tau' \quad \text{and} \quad \mathbb{B}_k = B_k\int_0^\tau \exp(b_k\tau')d\tau',$$

with $a_k=-(F_k-F_{k-1})$ and $b_k=-(F_k-F_{k-2})$. We now consider ordered sequences of knots obtained applying the following rules: (a) fix the order of the moment $n\in\{1,\ldots,N\}$; (b) fix $j\in\{1,\ldots,n\}$; (c) choose the first knot between $\mathbb{A}_n$ or $\mathbb{B}_n$; (d) move rightward adding a new knot. $\mathbb{A}_k$ can be followed by either $\mathbb{A}_{k-1}$ or by $\mathbb{B}_{k-1}$, while $\mathbb{B}_k$ can be followed by either $\mathbb{A}_{k-2}$ or $\mathbb{B}_{k-2}$; (e) if N_A and N_B are the number of type A and type B knots, respectively, stop when $N_A+2N_B=j$. We associate to every ordered sequence generated going through the previous procedure an integral made of N_A+N_B nested integrals. For instance, for $j=1$ the only admissible sequence is $\mathbb{A}_n$. For $j=2$, beside $\mathbb{B}_n$, we have to consider the sequence $\mathbb{A}_n\mathbb{A}_{n-1}$ giving the contribution

$$\mathbb{A}_n\mathbb{A}_{n-1} = A_nA_{n-1}\int_0^\tau \exp(a_n\tau_1)\int_0^{\tau_1} \exp(a_{n-1}\tau_2)d\tau_2 d\tau_1.$$

When $j=4$ the following strings have to be taken into account

$$\mathbb{A}_n\mathbb{A}_{n-1}\mathbb{A}_{n-2}\mathbb{A}_{n-3}, \quad \mathbb{A}_n\mathbb{A}_{n-1}\mathbb{B}_{n-2}, \quad \mathbb{A}_n\mathbb{B}_{n-1}\mathbb{A}_{n-3},$$

$$\mathbb{B}_n\mathbb{A}_{n-2}\mathbb{A}_{n-3}, \quad \mathbb{B}_n\mathbb{B}_{n-2}.$$

The special case $j=0$ is associated to a sequence with no knot whose contribution is equal to 1. Once n has been fixed, it is readily proved that every sequence is univocally determined retaining the label of the vertex while dropping the indexes. For the case $j=4$ above strings reduce to $\mathbb{AAAA},\mathbb{AAB},\mathbb{ABA},\mathbb{BAA},\mathbb{BB}$. We call $\Pi_{N_AN_B}$ the set of permutations with no repetition of N_A type A elements and N_B type B elements and $\pi_{N_AN_B}$ its generic element; the compact notation $\Delta_n[\pi_{N_AN_B},\tau(t)]$ identifies the N_A+N_B-dimensional integral contributing to the n-th moment and corresponding to the sequence of knots sorted according to $\pi_{N_AN_B}$. In terms of the above quantities, the expression of $\mu_n(t)$ can be usefully rewritten in the compact form

$$\exp[F_n\tau(t)]\sum_{j=0}^{n}\langle X_0^{n-j}\rangle \sum_{N_A+2N_B=j}\sum_{\Pi_{N_AN_B}}\Delta_n[\pi_{N_AN_B},\tau(t)]. \quad (5)$$

A careful analysis of the quantity $\Delta_n[\pi_{N_AN_B},\tau(t)]$ shows that it can always be computed analytically in an algorithmic way [15], which makes the expansion (5) a powerful tool to exactly compute μ_n up to an arbitrary order. Supposing that all the a_k and b_k involved in the expression of μ_n are non vanishing [16], the expansion (5) can be rewritten as

$$\mu_n(t) = \sum_{j=0}^{n} c_j^n \exp[F_{n-j}\tau(t)], \quad (6)$$

the c_j^n being real possibly vanishing functions of the A_k's, B_k's, F_k's, and $\langle X_0^k\rangle$. Above equation provides evidence of the typical scaling of the moments over time. The multiple time scales emerging from the multiplicative noise process can be affected by varying the functional form of $g(t)$. For a constant $g(t)=1$ [1], we have $\tau=t-t_0$ and the n-th order moment is characterized by the superposition of n exponentials with time constants $\{1/|F_n|,\ldots,1/|F_1|\}$. When $g(t)=t$, as in [10], we have terms of the form

$$\exp[F_{n-j}\tau(t)] = t^{F_{n-j}}t_0^{-F_{n-j}},$$

producing a power law time scaling of the moments. More generally, for $g(t)=t^\beta$ $(\beta\neq 1)$ the time dependence turns out to be a stretched exponential with stretching exponent $1-\beta$

$$\exp[F_{n-j}\tau(t)] = \exp[F_{n-j}(t^{1-\beta}-t_0^{1-\beta})/(1-\beta)].$$

Equation (6) also allows us to gain insight into the nature of the stochastic process described by the model (1), both at finite t and for t approaching $t_{\lim}$. As far as the PDF $p(x,t)$ associated to above process is concerned, in general we are not allowed to draw any conclusion about its shape. However, an important exception is the case when, for $t\to t_{\lim}$, we have a diverging τ. Indeed, in terms of τ the PDF satisfies the FP equation

$$\frac{\partial}{\partial\tau}\tilde{p}(x,\tau) = -\frac{\partial}{\partial x}[D_1(x)\tilde{p}(x,\tau)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D_2(x)\tilde{p}(x,\tau)], \quad (7)$$

with $D_1(x)=ax+b$ and $D_2(x)=cx^2+dx+e$ and initial condition $\tilde{p}(x,0)=\tilde{p}_0(x)$, for which the stationary solution $\tilde{p}_{st}(x)$ can be computed analytically following [1,11]. This solution provides evidence of the possible emergence of power law tails and Eq. (6) precisely characterizes the way the moments converge to the stationary level. Indeed, the smoothness of τ as a function of t implies $\lim_{t\to t_{\lim}^-}p(x,t)=\lim_{\tau\to+\infty}\tilde{p}(x,\tau)$ $=\tilde{p}_{st}(x)$. For example, if $g(t)=(t-t_{\lim})^\beta$ with $\beta>1$, the behavior of $\tilde{p}_{st}(x)$ emerging from the analysis of Eq. (7) applies to $p(x,t)$ when t approaches $t_{\lim}$, while the moments scale according to

$$\exp[F_{n-j}\tau(t)] = \exp(F_{n-j}[(t_0-t_{\lim})^{1-\beta}-(t-t_{\lim})^{1-\beta}]/(1-\beta)).$$

We now discuss how our results can be employed to characterize the stochastic process described by the SDE (1) for different choices of a, b, c, d, and e. Our analysis is essentially based on the sign of the factors F_{n-j} appearing in Eq. (6). Indeed, F_n is a convex function of n, depending only on a and c, whose zeros are $n_0=0$ and $n_1=1-2a/c\in\mathbb{R}$. If $n_1<0$ all the moments diverge when $t\to t_{\lim}^-$, while if $n_1>0$, all μ_n's for $n<n_1$ are convergent, otherwise not. For a convergent μ_n, the estimate of the rate of convergence is provided by $\tau_{\max}=\max(1/|a|,1/|F_n|)$ which corresponds to the largest relaxation time in Eq. (6). The cases $n_1=0,1,2$ have to be considered carefully, since they correspond to

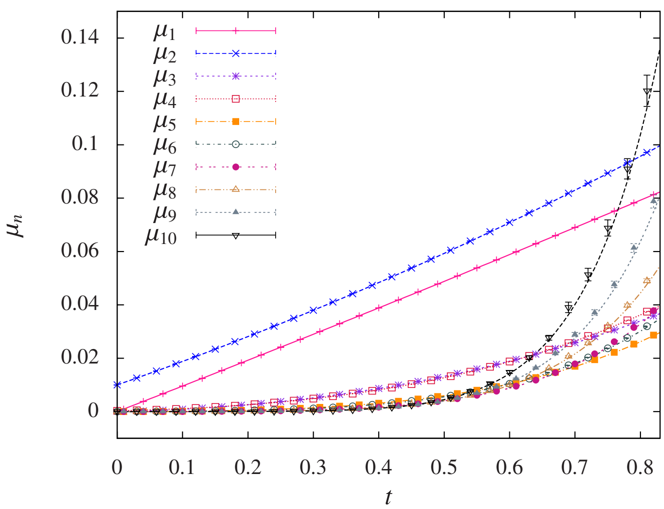

FIG. 1. (Color online) Scaling of the moments for $a=b=9.5\times10^{-2}$ and $c=d=e=8.3\times10^{-2}$; $\tilde{p}_0(x)$ is a zero mean Gaussian with $\langle X_0^2\rangle=0.01$. The Monte Carlo estimates of the μ_n's within 68% error bars are superimposed to the analytical curves, exhibiting full agreement.

$a=c/2$, 0 and $-c/2$, respectively, for which Eq. (6) does not apply (see [16]).

Case $a>0$ and $c>0$. All the moments diverge, since $F_n>0$ $\forall n>0$. From [1] the stationary solution can be defined only if $0<a<c/2$ and $e>0$ and it is a generalized Student t with tail exponent $\nu=1+n_1$. For $a\geq c/2$ and finite t all the moments are well-defined but no conclusions can be drawn about the exact form of the PDF. An example of the latter case is shown in Fig. 1, with all the μ_n diverging for large t.

Case $a=0$ and $e>0$. If $c=0$, then also d is 0 and Eq. (1) describes an Arithmetic Brownian motion with time dependent coefficients. If $c>0$, then $a_1=F_1=0$ and $\mathbb{A}_1=b\tau$, but integration by parts reveals that no moment converge, while the stationary solution is a power law with tail exponent $\nu=2$.

Case $a<0$, $c>0$, and $e>0$. $F_n>0$ for $n>n_1$, thus only the first $n<n_1$ moments converge to a stationary level. The special case $a=-c/2$ implies $n_1=2$, $b_2=F_2=0$ and $\mathbb{B}_2=e\tau$ and previous conclusions are unchanged. Coherently the solution of the FP equation predicts the emergence of power law tails with $\nu=1+n_1$. In Fig. 2 a case corresponding to $n_1\simeq9.9$ is shown for a stretched exponential time scaling, while Fig. 3 corresponds to the case $g(t)=(t-t_{\text{lim}})^2$.

Case $a\neq0$, $c=0$, and $e>0$. Equation (1) describes an Ornstein-Uhlenbeck process. F_n becomes a linear function of n and the moments reach a stationary value only if $a<0$. For $a>0$ the Gaussian PDF has time dependent unbounded mean and variance.

Case $e=0$ and $c>0$. As above the boundedness of the moments can be deduced from the value of n_1 and for $a<0$ the stationary solution is an Inverted Gamma with shape parameter $n_1>0$ and scale parameter $2|b|/c>0$. If $b>0$ the Inverted Gamma is defined for $x\in[0,+\infty)$, while for $b<0$ the support is $(-\infty,0]$. A similar situation occurs for $d^2-4ce=0$ and $d>0$, $c>0$, and $e>0$, but the support boundary point corresponds to $-d/(2c)$.

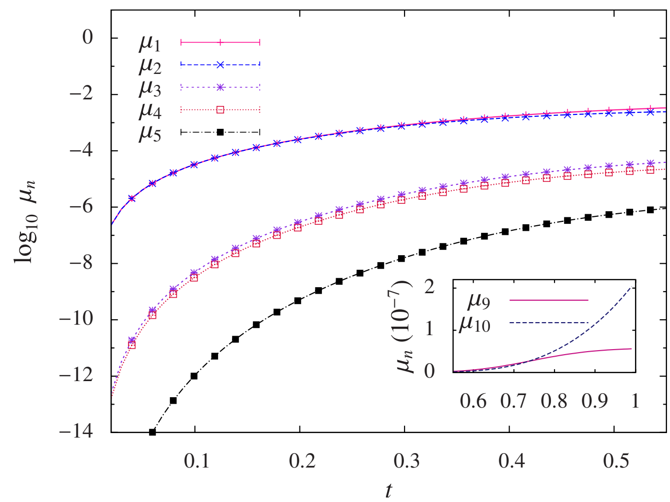

FIG. 2. (Color online) Lowest-order moments for $a=-20$, $b=d=e=0.1$, $c=4.5$, with $g=t^\beta$, $\beta=2$ and $\tilde{p}_0(x)=\delta(x)$. In the inset the last converging moment is compared to the first diverging one.

We can also deal with the more general case of a time-dependent $e(t)$, for which an analysis of the $\tilde{p}_{\text{st}}(x)$ cannot be performed straightforwardly. Here we outline how to proceed for two cases discussed in the literature [7,12], emerging in the context of financial time series analysis. Equation (6) in [12] leads to a SDE belonging to the class (1) for $g(t)=1$, implying $\tau=t-t_0$, $a=-4.4\times10^{-1}$, $b=0$, $c=3.8\times10^{-2}$, $d=3.04\times10^{-3}$, and $e(t)=e+e'e^{\epsilon t}$, with $e=6.08\times10^{-5}$, $e'=3\times10^{-3}$, and $\epsilon=-0.5$. The type B knot now splits into the sum of two contributions

$$\mathbb{B}_k=B_k\int_0^\tau \exp(b_k\tau')d\tau' \quad \text{and} \quad \mathbb{B}'_k=B'_k\int_0^\tau \exp(b'_k\tau')d\tau',$$

where $B'_k=\frac{1}{2}k(k-1)e'\exp(\epsilon t_0)$ and $b'_k=b_k+\epsilon$. The last two sums in Eq. (5) have to be coherently modified as

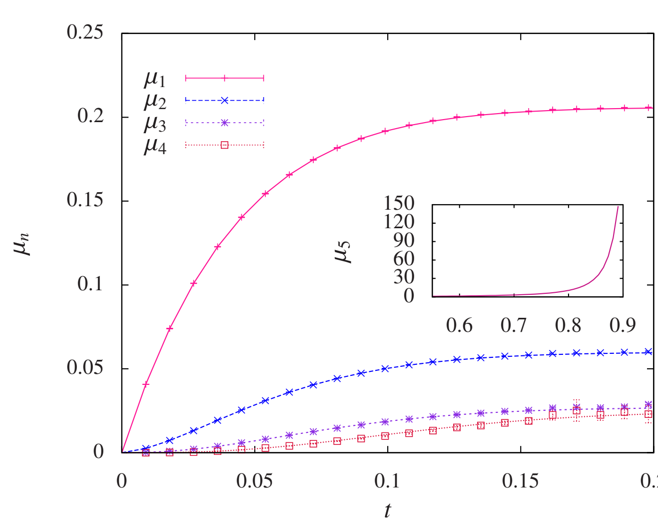

FIG. 3. (Color online) Lowest-order moments for $a=-24.3$, $b=5$, $c=12.2$, $d=e=0.1$, with $t_{\text{lim}}=1$ and $\tilde{p}_0(x)=\delta(x)$. The first diverging moment μ_5 is shown in the inset.

$$\sum_{N_A+2(N_B+N_{B'})=j} \sum_{\Pi_{N_A N_B N_{B'}}} \Delta_n[\pi_{N_A N_B N_{B'}}, t-t_0],$$

while Eq. (6) becomes

$$\mu_n(t) = \sum_{j=0}^{n} c_j^n(\epsilon, t_0)\exp[(F_{n-j} + d_j^n \epsilon)(t-t_0)],$$

with $d_j^n \in \mathbb{N}$.

The second case we want to discuss is the model assumed in [7] to describe the financial returns dynamics under the objective probability measure. It is readily proved that, defining $X_t = \ln S_t/S_0 - \mu t$, Eq. (2) in [7] corresponds to Eq. (1) for $g(t)=t$, $a=b=d=0$, $c=(q-1)/[(2-q)(3-q)]$, $e(t)=et^{2/(3-q)}$ with $e=\sigma^2[c(2-q)(3-q)]^{(q-1)/(3-q))}$, and $\langle X_0^n\rangle = 0$, $\forall n \geq 1$; σ^2 and c are positive constants, while the Tsallis entropic index q belongs to (1,5/3) to ensure the existence of mean and variance [13]. The starting time $t_0=0$ is a singular point for $1/g(t)$ and we reproduce the correct results in the limit $t_0 \to 0^+$. Since $\langle X_0^n\rangle = 0$ the only j contributing to the sum in Eq. (5) is $j=n$, while $b=d=0$ implies that every type A knot is identically zero and so are all the odd moments. Thus, expansion (5) reduces to $\exp[F_n \tau(t)]\Delta_n[\pi_{0n/2}, \tau(t)]$ with $\tau = \ln t - \ln t_0$ and $n=2p$, $p>1$. Now the type B knot reads

$$\mathbb{B}_k = \frac{1}{2}k(k-1)e \exp[2/(3-q) \ln t_0] \times \int_0^{\tau} \exp([b_k + 2/(3-q)]\tau')d\tau'.$$

The same analysis leading to Eq. (6) allows us to conclude that

$$\mu_{2p}(t) = \sum_{j=0}^{p} c_j^{2p} t^{F_{2(p-j)}+2j/(3-q)} t_0^{-[F_{2(p-j)}+2(j-p)/(3-q)]},$$

where the c_j^{2p} are all nonvanishing constants. The expression in the square brackets governs the limiting behavior of μ_{2p} when $t_0 \to 0^+$. As a function of j, $F_{2(p-j)}+2(j-p)/(3-q)$ is convex and for $j=p$ its value is zero, so that we only have to check the behavior for $j=0$. Indeed, if $F_{2p}+2p/(q-3)>0$ then $\lim_{t_0\to 0^+}\mu_{2p}=+\infty$, otherwise all the exponents of t_0 are non-negative and no divergence is possible. But recalling the expression of c, we find $q>(2p+3)/(2p+1)$, recovering the condition required in [7] to obtain a divergent $2p$-th order moment. It is worth noticing that the previous conclusions can be readily rephrased in terms of the Hurst exponent $H=1/(3-q)$ as it has been done in [5].

In conclusion, the processes described by Eq. (1) emerge in the study of many complex systems. An explicit expression for the associated PDF has not, however, been available for finite times. In this Brief Report, we provided a characterization in terms of the moments, deriving closed-form expressions at all orders and for all times. These results, revealing a simple relationship between the time dependence of Eq. (1) and the scaling behavior of the moments provides a better understanding of how these processes evolve with time, possibly but not necessarily toward a stationary state and provide a closed-form relation between the model parameters and their relaxation times. We also believe that these results can improve the statistical analysis of historical time series. Indeed, the analytical expressions allow to directly fit the time scaling of the empirical moments, providing a clear and simple way to fix the model parameters. Moreover, the knowledge of the moments is crucial to exploit analytical approximation to the full PDF associated to Eq. (1), such as the Edgeworth or more general expansions [14], even though recovering the PDF is a problem whose solution is in general not unique.

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[15] We will detail the derivation of Eqs. (5) and (6), as well as the algorithm allowing their numerical evaluation, in a forthcoming paper.
[16] We do not consider in this work the case for an integer or semi-integer ratio between $|a|$ and c. This case can be dealt with by means of nested integrations by parts.